Chiral solitons from dimensional reduction of Chern-Simons gauged non-linear Schrödinger model of FQHE: classical and quantum aspects*

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The soliton structure of a gauge theory recently proposed to describe chiral excitations in the Fractional Quantum Hall Effect is investigated. A new type of non-linear derivative Schrödinger equation emerges as an effective description of the system that supports novel *chiral* solitons. We discuss the classical properties of solutions with vanishing and non-vanishing boundary conditions (dark solitons) and we explain their relation to integrable systems. The quantum analysis is also addressed in the framework of a semiclassical approximation improved by Renormalization Group arguments.

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I. INTRODUCTION

One of the most fascinating aspects of classical field theory is the existence of travelling localized solutions of the non-linear equations describing physical systems: these *solitons* find application in a very broad class of physical problems. From the quantum field theory point of view, (being exact non-perturbative solutions of the classical theory) they are believed to carry information about the non-perturbative structure of the quantum theory. Their particle-like properties have an intuitive and reasonable interpretation as bound states of the elementary excitations of the corresponding quantum field theory. For specific two-dimensional models this idea was confirmed by explicit computations in the framework of the semiclassical approximation [1], of quantum inverse scattering [2] and of other systematic expansions [3] while, in more recent years, conformal field theory techniques have led to exact results for the $S-$matrix [4]. Solitons also play fundamental role in recent non-perturbative developments of QFT (for a recent review, see [5]), since they are seen as the "duals" of the elementary excitations appearing in the original Lagrangian [6].

Solitons made their appearance in a completely different context some years ago, namely in low-energy phenomenological applications to physical systems confined to a plane. An interesting class of gauge theoretical models, describing non-relativistic matter coupled to a Chern-Simons gauge field, was introduced by Jackiw and Pi [7] to obtain a simple realization of non-relativistic interacting anyons. For an appropriate choice of the self-interaction potential, in the static case, one can reduce the Euler-Lagrange equations to the completely integrable Toda equation (in the non-Abelian situation) or Liouville (in the Abelian one) [8], with well-known soliton solutions. The dynamics of the system corresponds, in this case, to a dimensional reduction in time: recently [9,10] there has been instead considered a reduction in one spatial dimension. The physical reason lies in the hope that some characteristic of the model, in particular its fractional statistics, can be maintained by the related one-dimensional excitations and that, by a suitable modification, chiral behaviour can be induced. These two features are in fact relevant in the phenomenological description of the edge states in the Fractional Quantum Hall Effect [11]. Unfortunately the theory obtained

in this way, originally proposed in [9], partially fails to achieve its goal, because as was shown in [10] no statistical transmutation arises for non-relativistic matter. On the other hand, as first observed in [10], a novel and interesting soliton structure is present there, finding its origin in the gauge coupling and in the chiral modification. Our investigation is a direct follow-up of the work begun in [10], and it is particularly devoted to the classical and quantum analysis of the solitons appearing in the theory. Actually, after realizing that the symmetry property of the original model is shared by a larger class of $1+1$ dimensional theories, we have chosen to study a slight generalization of the model of [9,10], that fits equally well into the dimensional reduction procedure. Surprisingly, as we shall see, quantum effects naturally lead one to include the more general interactions that we have introduced. As in the analogous $2+1$-dimensional family, integrability appears only for a suitable choice of the initial parameters (as observed in [12]), relating our system to the Derivative Non-Linear Schrödinger equation (DNLS) [13]. Nevertheless soliton solutions exist for any value of the coupling constants.

We start in Sect. 2 by describing the dimensional reduction of the Jackiw-Pi model and its modification, based essentially on the introduction of a chiral boson. The gauge action turns out to be of $B-F$ type and we show that the system is equivalent to a general family of non-linear Schrödinger equations that do not possess Galilean invariance. In absence of a self-interaction potential for the matter, the model has non-relativistic scale invariance, an important ingredient in understanding some features of the theory: a suitable potential respecting the scale invariance can also be added. We discuss the conserved charges and we derive, from symmetry considerations, some general properties of localized classical solution. In Sect. 3 we compute the solitons, first without and then including a potential, and we discuss their properties: in particular they appear to be chiral, existing only for a fixed sign of the total momentum, and presenting an interesting particle-like behaviour, inherited from the scale invariance. We also consider finite energy solutions with non-vanishing boundary condition at (spatial) infinity (dark solitons), that possess opposite chirality with respect to the previous ones. We show that conserved charges must be carefully defined when non-trivial boundary conditions are present and we derive the relevant energy-momentum

dispersion relation. In Sect.4, we discuss the quantization of the theory, paying particular attention to the relation between the solitons and the quantum bound states. Since the system is non-integrable for a generic choice of the coupling constants, it is difficult to obtain exact results, and we have explicitly solved the Schrödinger equation only for the two-body problem. Nevertheless we have obtained a perturbative solution, consistent with the Bohr-Sommerfield quantization of the solitons (in the weak-coupling limit) and the exact expression of the trace anomaly of the theory, showing that classical scale invariance is destroyed, except at the fixed point of the renormalization group, represented by DNLS. Combining this result with the semiclassical quantization of the soliton, we conjecture the general form of the energy for the n-body bound-state. Finally , in Sect. 5 we present our conclusion and indicate future directions.

II. NON-LINEAR DERIVATIVE SCHRÖDINGER EQUATIONS FROM A GAUGE THEORY AND THEIR CLASSICAL SYMMETRIES

A nonrelativistic gauge field theory that leads to planar anyons [7] is the nonlinear Schrödinger equation, gauged by a Chern-Simons field and governed by the Lagrange density

$$\mathcal{L}_{(2+1)} = \frac{1}{4\bar{\kappa}}\epsilon^{\alpha\beta\gamma}\hat{A}_\alpha \hat{F}_{\beta\gamma} + i\hbar\Psi^*(\partial_t + i\hat{A}_0)\Psi - \frac{\hbar^2}{2m}\sum_{i=1}^{2}\left|(\partial_i + i\hat{A}_i)\Psi\right|^2 - V(\Psi^*\Psi) \qquad (1)$$

Here Ψ is the Schrödinger quantum field, giving rise to charged bosonic particles after second quantization. $\hat{A}_\mu$ possesses no propagating degrees of freedom; it can be eliminated, leaving a statistical Aharonov-Bohm interaction between the particles. $V(\Psi^*\Psi)$ describes possible nonlinear self-interactions, and can be a general polynomial in the density $\Psi^*\Psi$. We notice that the above Lagrangian is invariant under Galilean transformations, due to the topological nature of the Chern-Simons action. When analyzing the lineal problem [10], it is natural to consider a dimensional reduction of (1), by suppressing dependence on the second spatial coordinate, renaming $\hat{A}_2$ as $(mc/\hbar^2)B$ and redefining the gauge field as $\hat{A}_x = A_x$ and $\hat{A}_0 = A_0 + mc^2/2\hbar B^2$. In this way one is led to a B–F gauge theory coupled to a non-relativistic bosonic field in $1+1$ dimensions:

$$\mathcal{L}_{(1+1)} = \frac{1}{2\kappa} B\epsilon^{\mu\nu} F_{\mu\nu} + i\hbar\Psi^* (\partial_t + iA_0)\Psi - \frac{\hbar^2}{2m} \left|(\partial_x + iA_x)\Psi\right|^2 - V(\Psi^*\Psi), \qquad (2)$$

where $\kappa \equiv (\hbar^2/mc)\bar{\kappa}$ is dimensionless and we have neglected $\partial_x(B^3/3\hbar\kappa)$ since it is a total spatial derivative. Eliminating the B and A_μ fields decouples them completely, in the sense that the phase of Ψ may be adjusted so that the interactions of the Ψ field are solely determined by V, and particle statistics remain unaffected [14]. In fact the equation of motion obtained by varying B, $F_{\mu\nu} = 0$, implies that the gauge field A_μ is a pure gauge and it can be reabsorbed in a phase redefinition of Ψ. In order to obtain a non-trivial theory even in absence of V it is quite natural to introduce a kinetic term for B, which, for example, could be taken to be in the Klein-Gordon form. In the following, however, we prefer a simpler expression that describes "chiral" Bose fields, propagating only in one direction. The reason why we commit ourselves to this specific choice is twofold: we hope both to induce a statistical transmutation and that the chiral dynamics is inherited by Ψ. Having a field theoretical model that was both chiral and with anomalous statistics would be of great relevance in the description of the edge states [11]. As we shall see, while this model achieves the first goal (chirality), it fails the other one. On the other hand its structure is so rich that a detailed study of its properties has intrinsic interest.

The Lagrangian density for the chiral boson is proportional to $\pm\dot{B}B' - vB'B'$ [15]. (Dot/prime indicate differentiation with respect to time/space.) Here v is a velocity and the ensuing equations of motion for this kinetic term (without further interaction) are solved by $B = B(x \pm vt)$ (with suitable boundary conditions at spatial infinity), describing propagation with velocity $\mp v$. Note that $\dot{B}B'$ is not invariant against a Galileo transformation, which is a symmetry of $\mathcal{L}_{(1+1)}$ and of $B'B'$: performing a Galileo boost on $\dot{B}B'$ with velocity $\tilde{v}$ gives rise to $\tilde{v}B'B'$, effectively boosting the v parameter by $\tilde{v}$. Consequently one can drop the $vB'B'$ contribution to the kinetic B Lagrangian, thereby selecting to work in a global "rest frame". Boosting a solution in this rest frame then produces a solution to the theory with a $B'B'$ term. In view of the above, we choose the B-kinetic Lagrange density to be

$$\mathcal{L}_B = \frac{\lambda}{2\kappa^2\hbar} \dot{B}B' \qquad (3)$$

and the total Lagrange density is $\mathcal{L} = \mathcal{L}_B + \mathcal{L}_{(1+1)}$. The ensuing equations of motion are

$$i\hbar(\partial_t + iA_0)\Psi + \frac{\hbar^2}{2m}(\partial_x + iA_1)^2\Psi - V'\Psi = 0, \tag{4a}$$

$$F_{01} - \frac{\lambda}{\kappa\hbar}\dot{B}' = 0, \tag{4b}$$

$$B' - \hbar\kappa\Psi^*\Psi = 0, \tag{4c}$$

$$\dot{B} + \hbar\kappa\hat{J} = 0, \tag{4d}$$

with $\hat{J} = \frac{\hbar}{2im}\Big(\Psi^*(\partial_x + iA_1)\Psi - \Psi(\partial_x - iA_1)\Psi^*\Big)$. The integrability condition for the last two equation leads to the usual continuity equation

$$\partial_t(\Psi^*\Psi) + \partial_x\hat{J} = 0. \tag{5}$$

Now in terms of field $\hat{\phi} = \exp\left(i\int_{x^0}^{x} dy\ A_1(y,t) + i\int_{t^0}^{t} dt'\ A_1(x^0,t') - i\frac{\lambda}{\kappa\hbar}B(x^0,t)\right)\Psi$ eqs (4a) and (4b) become respectively

$$i\hbar\partial_t\hat{\phi} + \frac{\hbar^2}{2m}\partial_x^2\hat{\phi} - \hbar\lambda j\hat{\phi} - V'\hat{\phi} = 0, \tag{6}$$

$$F_{01} = \lambda\partial_t(\hat{\phi}^*\hat{\phi}). \tag{7}$$

The latter gives the electromagnetic field as a function of the density $\hat{\phi}^*\hat{\phi}$ while the former encodes all the dynamical content of the system. In the following we shall be mainly interested in the case in which the potential $V(\hat{\phi}^*\hat{\phi})$ is absent: this corresponds to a Schrödinger equation with a *current* (j) nonlinearity:

$$j = \frac{\hbar}{2im}\left(\hat{\phi}^*\partial_x\hat{\phi} - \hat{\phi}\partial_x\hat{\phi}^*\right). \tag{8}$$

This is to be contrasted with the familiar nonlinear Schrödinger equation, where the nonlinearity involves the *charge density* ($\hat{\phi}\hat{\phi}^*$):

$$i\hbar\partial_t\hat{\phi} = -\frac{\hbar^2}{2m}\partial_x^2\hat{\phi} - \lambda(\hat{\phi}\hat{\phi}^*)\hat{\phi}. \tag{9}$$

Note that the gauge interaction has dynamically produced a non-trivial alternative to the usual non-linear Schrödinger equation, much studied both from physical and mathematical points of view. However eq. (6), in contrast to eq. (9), does not possess a local Lagrangian formulation directly in terms of the field $\hat{\phi}$. A local Lagrangian can be instead exhibited for the equation

$$i\hbar\partial_t\phi = -\frac{\hbar^2}{2m}\left(\partial_x + i\frac{\lambda}{2}\rho^2\right)^2\phi + \frac{\lambda\hbar}{2}J\phi + V'\phi, \tag{10}$$

governing the gauge equivalent variable $\phi = \exp\left[i\frac{\lambda}{2}\left(\int_{x^0}^{x}dy(\hat{\phi}^*\hat{\phi})(y,t) - \int_{t^0}^{t}dt' j(x^0,t')\right)\right]\hat{\phi}$. Consider, in fact, the action

$$S = \int dt\,dx\mathcal{L} = \int dt\,dx\left[\frac{i\hbar}{2}(\hat{\phi}^*\partial_t\phi - \phi\partial_t\phi^*) - \frac{\hbar^2}{2m}\left|\left(\partial_x + i\frac{\lambda}{2}\rho^2\right)\phi\right|^2 - V(\rho^2)\right]; \tag{11}$$

the Euler-Lagrange equations that follow can be easily shown to reproduce eq. (10). In eqs. (10) and (11) ρ^2 represents the density $\phi^*\phi$, while

$$J = \frac{\hbar}{2im}\left(\phi^*\left(\partial_x + i\frac{\lambda}{2}\rho^2\right)\phi - \hat{\phi}\left(\partial_x - i\frac{\lambda}{2}\rho^2\right)\phi^*\right) \tag{12}$$

is the corresponding current.

The invariance of the action S under space/time translations reflects itself into the presence of a divergenceless energy momentum tensor $T^{\lambda}_{\ \mu}$:

$$T^0_{\ 0} = \mathcal{H} = \left[\frac{\hbar^2}{2m}|D_x\phi|^2 + V(\rho^2)\right] \qquad T^x_{\ 0} = -\frac{\hbar^2}{2m}\left[D_x\phi\partial_t\phi^* + (D_x\phi)^*\partial_t\phi\right], \tag{13a}$$

$$T^0_{\ x} = \mathcal{P} = mJ - \hbar\frac{\lambda}{2}\rho^4 \qquad T^x_{\ x} = \frac{\hbar^2}{m}|D_x\phi|^2 - \frac{\hbar^2}{4m}\partial_x^2\rho^2 + V'\rho^2 - V, \tag{13b}$$

where D_x stands for the "covariant" derivative $\partial_x + i\lambda/2\rho^2$. For solutions obeying suitable boundary conditions, we can therefore write a conserved momentum $P = \int dx\ \mathcal{P}$ and energy $E = \int dx\ \mathcal{H}$. The unconventional form of the momentum density signals lack of Galilean invariance. In fact, in contrast to the Galilean invariant case, $\mathcal{P}$ is not proportional to the $U(1)$ current J. Indeed computing the time derivative of the Galileo generator $G = tP - m\int dxx\rho^2$, we find

$$\frac{dG}{dt} = \int dx(\mathcal{P} - mJ) = -\hbar\frac{\lambda}{2}\int dx\rho^4, \tag{14}$$

namely G, depending on the sign of the coupling constant, always increases or decreases in time. On the other hand, for a particular choice of the potential V, the theory becomes scale invariant. In fact the action (11) is unchanged under a dilation, $t \to a^2 t,\ x \to ax$, and

$\phi(x,t) \to a^{\frac{1}{2}}\phi(a^2 t, ax)$, iff $V(\rho^2/a)a^3 = V(\rho^2)$: $V(\rho^2) \propto \rho^6$. The generator D of the scale symmetry takes the form

$$D = \int dx \mathcal{D} = tH - \frac{1}{2}\int dx x \mathcal{P}, \tag{15}$$

where the density $\mathcal{D} = tT^0_{\ 0} - \frac{1}{2}xT^0_{\ x}$ obeys the continuity equation

$$\partial_t \mathcal{D} + \partial_x \left(t\, T^x_{\ 0} - \frac{1}{2}x\, T^x_{\ x} - \frac{\hbar^2}{8m}\partial_x \rho^2 \right) = 0. \tag{16}$$

We can remove the last term in eq. (16) proportional to the derivative of ρ^2 by adding a superpotential to the energy-momentum tensor. In fact if we define an improved $\hat{T}^\lambda_{\ \mu}$

$$\hat{T}^0_{\ 0} = T^0_{\ 0} - \frac{\hbar^2}{8m}\partial^2_x \rho^2, \qquad \hat{T}^x_{\ 0} = T^x_0 - \frac{\hbar^2}{8m}\partial^2_x J, \qquad \hat{T}^0_{\ x} = T^0_{\ x}, \qquad \hat{T}^x_{\ x} = T^x_{\ x}, \tag{17}$$

we obtain

$$\partial_t \mathcal{D} + \partial_x \left(t\hat{T}^x_{\ 0} - \frac{1}{2}x\hat{T}^x_{\ x} \right) = 0. \tag{18}$$

The new energy momentum tensor satisfies $2\hat{T}^0_{\ 0} = \hat{T}^x_{\ x}$, which is the non relativistic analog of the relativistic traceless condition for scale symmetry. In a Galilean invariant theory this condition would also ensure conformal invariance; here, instead, the absence of Galilean symmetry requires the absence of conformal symmetry. Notice that the classical symmetries of our system form a three dimensional Lie sub-algebra of the conformal group (non-relativistic Weyl group in $(1+1)-$dimensions) with Poisson brackets

$$\{H, P\} = 0, \qquad \{D, P\} = P, \qquad \{D, H\} = 2H. \tag{19}$$

In the following we shall be interested in classical "localized" solutions of the scale invariant Hamiltonian

$$H = \frac{\hbar^2}{2m}\int dx \left[\left| \left(\partial_x + i\frac{\lambda}{2}\rho^2 \right)\phi \right|^2 - \frac{\lambda^2 \xi}{4}(\phi\phi^*)^3 \right], \tag{20}$$

namely normalizable field configurations ($N = \int dx \rho^2 < \infty$) with finite energy and momentum. Here ξ is a dimensionless parameter governing the strength of the cubic potential. In the absence of V (*i.e.* $\xi = 0$) we recover the model of ref. [9,10] that will be our main

concern and whose solutions we shall present in detail. Some results about $\xi \neq 0$ will be also discussed as well because of their relevance in the quantum theory.

Unlike the conventional nonlinear Schrödinger equation, the family of scale invariant equations we have described, and in particular the one without cubic potential, does not appear to be completely integrable and thus analytic expressions for multisoliton solutions are not available. It has been remarked, however, that for $\xi = 1$ the resulting Schrödinger equation becomes an integrable nonlinear, derivative Schrödinger equation with nonlinearity $i\frac{\hbar^2\lambda^2}{2m}\rho\partial_x\hat{\phi}$ [12].

III. CLASSICAL SOLUTIONS

When it comes to the problem of solving non-linear differential equations, very few general tools are available. One possibility is to look for solutions that possess particular symmetries or whose specific functional dependence simplifies the structure of the original equation. In our case a simple Ansatz, which allows us to integrate eq. (10), is to assume that the density is a function only of $x - vt$, i.e. $\rho(x,t) \equiv \rho(x - vt)$. Drawing from the experience of integrable models, we can expect that this choice will allow us to explore the presence of one-soliton solutions or more generally of traveling waves. Substituting the ansatz into the continuity equation, yields

$$\partial_x(-v\rho^2(x - vt) + J(x,t)) = 0, \tag{21}$$

and hence

$$J(x,t) = v\rho^2(x - vt) + J_\infty(t). \tag{22}$$

Here $J_\infty(t)$ is an arbitrary function of time, whose physical meaning will become transparent later. [It is already clear that such a function is related to the boundary conditions chosen at infinity.] In the following we shall be concerned with solutions that approach the vacuum at spatial infinity. Since $\mathcal{H}$ is positive definite in the absence of the potential $V(\rho^2)$, the vacuum solutions is first constrained by the requirement

$$H = 0 \qquad \Longrightarrow \qquad \Pi \equiv \left(\partial_x + i\frac{\lambda}{2}\rho^2\right)\phi = 0, \tag{23}$$

which is solved by taking ϕ of the form

$$\phi(x,t) = \rho_0(t)\exp\left(-i\omega(t) - i\frac{\lambda}{2}\rho_0^2(t)x\right). \tag{24}$$

Requiring (24) to satisfy the equation implies that $\rho_0(t)$ and $\omega(t)$ must be constants. Thus

$$\hat{\phi}(x,t)_{vacuum} = \rho_0 \exp\left[-i\left(\frac{\lambda}{2}\rho_0^2 x + \theta_0\right)\right]. \tag{25}$$

Let us notice that the vacuum solution (25) is characterized by a vanishing current $J(x,t)$. Thus requiring that a solution approach the vacuum unambiguously fixes the value of $J_\infty(t)$ in eq. (22), namely

$$J(x,t) = v(\rho^2(x-vt) - \rho_0^2). \tag{26}$$

This brief analysis of the space of vacua suggests that we distinguish two possible cases: solutions that approach the trivial vacuum $\phi_{vacuum} = 0$ ($\rho_0 = 0$) at spatial infinity, and the solutions that, instead, go to the ground state that has constant density $\rho_0 \neq 0$. The former are characterized by well-defined energy, momentum and $U(1)$ charge. The latter have finite energy, but infinite momentum and $U(1)$ charge. This different behavior is easily understood in terms of the properties of the vacuum. In fact, while in both cases the vacuum is is invariant under time translation, the space translation and $U(1)$ transformation leave it invariant only if $\rho_0 = 0$.

A. Solutions around the trivial vacuum ($\rho_0 = 0$)

This class of solutions is strongly constrained by the symmetry of the problem. Let us show how a wide number of their properties can be derived without explicitly solving the equation of motion. To begin with, the simple Ansatz $\rho \equiv \rho(x - vt)$, by means of eq. (26), implies that the momentum density $\mathcal{P}$ is a function of $x - vt$ as well. Thus the dilation charge takes the form

$$D = tH - \frac{1}{2}\int_{-\infty}^{\infty} dx(x-vt)\mathcal{P}(x-vt) - \frac{v}{2}t\int_{\infty}^{\infty} dx\mathcal{P}(x-vt) = t(H - \frac{v}{2}P) - \frac{1}{2}D_0, \tag{27}$$

where $D_0 = \int_{-\infty}^{\infty} dx x\mathcal{P}(x)$. Since D is conserved and consequently time-independent we obtain

$$H = \frac{v}{2}P, \tag{28}$$

namely the dispersion relation of a non-relativistic particle. Note that the result holds independently of the value of ξ. In fact this reasoning does not make use of the explicit form of the Hamiltonian, but only of its symmetries.

To study further properties, we introduce the "center of mass" coordinate

$$x_{CM}(t) = \frac{\int_{-\infty}^{\infty} dx\ x\rho^2(x,t)}{\int_{-\infty}^{\infty} dx\ \rho^2(x,t)}. \tag{29}$$

This name is easily understood if we think of ρ^2 as the mass density. Its velocity will be

$$v_{CM} = \dot{x}_{CM}(t) = \frac{\int_{-\infty}^{\infty} dx\ J(x,t)}{N}, \qquad \text{with} \quad N \equiv \int_{-\infty}^{\infty} dx\ \rho^2(x,t). \tag{30}$$

Here we have used the continuity equation for the current to eliminate the time derivative of the density. In this language the violation of Galilean invariance (14) takes a suggestive form,

$$\lambda(P - mNv_{CM}(t)) = -\hbar\frac{\lambda^2}{2}\int_{-\infty}^{\infty} dx\ \rho^4(x,t) \leq 0. \tag{31}$$

Being valid for all t, this implies

$$\lambda P \leq mN\lambda \min_{t\in \mathrm{R}}\{v_{CM}(t)\} \tag{32}$$

For $\xi < 0$ (*i.e.* repulsive potential and thus energy positive definite), one can show an analogous inequality for the energy. In fact let us consider the following inequality

$$\int_{-\infty}^{\infty} dx \left|\phi + w\frac{\hbar}{2mi}D_x\phi\right|^2 \geq 0, \tag{33}$$

where w is an arbitrary parameter. In terms of the physical quantities

$$N + wNv_{CM}(t) + w^2\frac{E_0}{2m} \geq 0. \tag{34}$$

with $E_0 = \frac{\hbar^2}{2m}\int_{-\infty}^{\infty} dx\,|D_x\phi|^2$. The fact that the previous equation holds for all w entails

$$E_0 \geq \frac{mNv_{CM}^2(t)}{2}, \tag{35}$$

but, for $\xi < 0$,the real energy H is greater than E_0 and thus

$$E \geq \frac{mNv_{CM}^2(t)}{2}. \tag{36}$$

Let us apply these inequalities to our Ansatz. A simple computation shows that $v_{cm}(t) = v$. Thus eq. (31) can be rewritten as

$$\lambda v\left(E - \frac{mNv^2}{2}\right) \leq 0, \tag{37}$$

where we used eq. (28). For $\xi < 0$, because of (36), it implies $\lambda v < 0$, *i.e.* the soliton is "chiral". In other words, given the sign of the coupling constant λ, the sign of the velocity is determined. Recalling that

$$P = mNv - \hbar\frac{\lambda}{2}\int_{-\infty}^{\infty}\rho^4(x,t)dx, \tag{38}$$

the chirality can be also written in the form $\lambda P < 0$. In this weaker form it is actually true for $\xi > 0$ as well (see below).

We return now to the problem of finding explicit solutions. Upon introducing the parametrization

$$\phi(x,t) = \rho(x - vt)\exp[i\theta(x,t)], \tag{39}$$

eq. (26) becomes

$$\partial_x\theta(x,t) = \frac{mv}{\hbar} - \frac{\lambda}{2}\rho^2(x - vt) \quad\Longrightarrow\quad \theta(x,t) = \frac{mv}{\hbar}x + \theta_0(t) - \frac{\lambda}{2}\int_{-\infty}^{x-vt} dy\;\rho^2(y), \tag{40}$$

Substituting (40) into the equation of motion, we obtain

$$\frac{2m}{\hbar}\dot\theta_0(t)\rho(y) = \rho''(y) - \frac{m^2v^2}{\hbar^2}\rho(y) - \frac{2mv}{\hbar}\lambda\rho^3(y). \tag{41}$$

Here $y \equiv x - vt$ and the prime denotes the derivative with respect to y. The consistency of eq. (41) requires that $\dot\theta_0(t)$ is a constant ω_0. Thus our equation can be rewritten as

$$\rho'' - \frac{2m}{\hbar}\left(\omega_0 + \frac{mv^2}{2\hbar}\right)\rho - \frac{2mv}{\hbar}\lambda\rho^3 = 0, \tag{42}$$

or equivalently

$$(\rho')^2 - \frac{m^2v^2}{4\hbar^2}\gamma\rho^2 - \frac{mv}{\hbar}\lambda\rho^4 = 0, \tag{43}$$

where $\frac{mv^2}{8\hbar}\gamma = \left(\omega_0 + \frac{mv^2}{2\hbar}\right)$. The possible arbitrary constant integration in eq. (43) is fixed by imposing that $\rho \to 0$ as $x \to \pm\infty$. The problem of finding normalizable solutions is thus reduced to computing the zero energy orbits for a particle moving in a effective quartic potential. It is well-known that non-trivial (*i.e.* not identically constant) zero-energy orbits exist if and only if

$$\lambda v < 0 \qquad \text{and} \qquad \gamma > 0. \tag{44}$$

The first condition is particularly intriguing: given the sign of the coupling constant, a solution can be found only for a given sign of v, namely the system is *"chiral"*. Integrating eq. (43) in the allowed parameter region, we obtain

$$\rho(x - vt) = \frac{1}{2}\sqrt{\left|\frac{mv}{\lambda\hbar}\right|\gamma}\ \mathrm{sech}\left[\frac{m|v|}{2\hbar}\sqrt{\gamma}(x - x_0 - vt)\right]. \tag{45}$$

The phase can be in turn computed with the help of eq. (40)

$$\theta = \frac{mv}{\hbar}x + \omega_0 t + \theta_0 - \frac{\sqrt{\gamma}}{4}\mathrm{sign}(\lambda)\tanh\left[\frac{m|v|}{2\hbar}\sqrt{\gamma}(x - x_0 - vt)\right]. \tag{46}$$

The soliton's dynamical parameters, such as the $U(1)$ charge, the energy or the momentum, can be now evaluted. Setting the $U(1)$ charge

$$N = \int_{-\infty}^{+\infty} dx\, \rho^2 = \frac{\sqrt{\gamma}}{|\lambda|}. \tag{47}$$

we find that the energy and the momentum of the solitonic solution are

$$E = \frac{1}{2}Mv^2, \quad \text{and} \quad P = Mv \quad \text{where} \quad M = mN(1 + \frac{1}{12}\lambda^2N^2). \tag{48}$$

The soliton's characteristics are those of a non-relativistic particle of mass M, moving with velocity v and composed of N "constituents". Notice that N is a function of v and ω_0: for a given N we can get solitons of arbitrary velocity simply by tuning the phase velocity.

As we have seen in the previous section many properties are shared by the whole class of scale invariant systems: it is therefore interesting to explore the one-soliton structure of our theory when a potential cubic in ρ^2 is present. We parametrized $V(\rho^2)$ as

$$V(\rho^2) = -\frac{\hbar^2\lambda^2\xi}{8m}\rho^6, \tag{49}$$

with $\xi \in (-\infty, +\infty)$. For $\xi = 1$ we recover the integrable model described by the non-linear derivative Schrodinger equation, studied in [12]. Taking into accounts vanishing boundary conditions for ρ we simply have

$$(\rho')^2 - \frac{m^2v^2}{4\hbar^2}\gamma\rho^2 - \frac{mv}{\hbar}\lambda\rho^4 + \frac{\lambda^2\xi}{4}\rho^6 = 0. \tag{50}$$

After defining $Z = \frac{\hbar^2\lambda^2|\xi|}{m^2v^2}\rho^2$ and eq. (50) becomes

$$\begin{aligned}(Z')^2 &= \frac{m^2v^2}{\hbar^2\xi}Z^2\left[(\gamma\xi+4) - (Z - 2\,\mathrm{sign}(\lambda v))^2\right] \qquad \xi > 0\\ (Z')^2 &= \frac{m^2v^2}{\hbar^2|\xi|}Z^2\left[(\gamma|\xi|-4) + (Z + 2\,\mathrm{sign}(\lambda v))^2\right] \qquad \xi < 0.\end{aligned} \tag{51}$$

Let us first consider $\xi > 0$: following the same arguments of $\xi = 0$ case, we have normalizable solution for $\gamma > 0$ that are

$$\rho^2 = |\frac{mv}{\lambda\hbar}|\frac{\gamma}{\sqrt{\gamma\xi+4}\cosh\left[\frac{mv}{\hbar}\sqrt{\gamma}(x-x_0-vt)\right] - 2\mathrm{sign}(\lambda v)}. \tag{52}$$

In this range the solitons are not chiral, solutions exist for both the signs of the velocity: the condition on ω_0 is exactly the same as when $\xi = 0$. Moreover it is immediately realized that the limit $\xi \to 0$ reproduces the correct solution only for $\lambda v < 0$, the other sign leading to a singular function. For $\xi < 0$ the situation is rather different: a short analysis of the "fictitious" potential yields the condition

$$\lambda v < 0 \qquad \text{and} \qquad 0 < \gamma < \frac{4}{|\xi|}. \tag{53}$$

The functional form of ρ^2 is nevertheless similar,

$$\rho^2 = |\frac{mv}{\lambda\hbar}|\frac{\gamma}{\sqrt{4-\gamma\xi}\cosh\left[\frac{mv}{\hbar}\sqrt{\gamma}(x-x_0-vt)\right] + 2}. \tag{54}$$

This family of solutions is chiral and reproduces the solitons of the original equation. The different features of the two ranges and the particularity of the point $\xi = 0$ are clearly displayed when the conserved quantities are computed:

$$N = \frac{4}{|\lambda|\sqrt{\xi}} \arctan \sqrt{\frac{\sqrt{4+\gamma\xi} + 2\ \mathrm{sign}(\lambda v)}{\sqrt{4+\gamma\xi} - 2\ \mathrm{sign}(\lambda v)}}, \tag{55a}$$

$$E = \frac{mv^2}{2|\lambda|} \left[\frac{(\xi - 1)}{\xi} N\lambda - \frac{2\ \mathrm{sign}(\lambda v)}{\sqrt{\xi^3}} \left| \tan\left(\frac{N\lambda\sqrt{\xi}}{2}\right) \right| \right], \tag{55b}$$

for $\xi > 0$ and

$$N = \frac{2}{|\lambda|\sqrt{\xi}} \log \left[\frac{1 + \sqrt{2 - \sqrt{4 - \gamma\xi}}}{1 - \sqrt{2 - \sqrt{4 - \gamma\xi}}} \right] \tag{56a}$$

$$E = \frac{mv^2}{2|\lambda|} \left[\frac{(1 + |\xi|)}{|\xi|} N|\lambda| - \frac{2}{\sqrt{|\xi|^3}} \tanh\left(\frac{N|\lambda|\sqrt{|\xi|}}{2}\right) \right], \tag{56b}$$

for $\xi < 0$. We observe that for $\xi > 0$ N takes value in a closed interval:

$$0 \leq N \leq \frac{\pi}{|\lambda|\sqrt{\xi}} \qquad \lambda v < 0, \tag{57a}$$

$$\frac{\pi}{|\lambda|\sqrt{\xi}} \leq N \leq \frac{2\pi}{|\lambda|\sqrt{\xi}} \qquad \lambda v > 0. \tag{57b}$$

No restriction arises for $\xi < 0$. The formulae for the energy and the number present a potential singularity at $\xi = 0$; actually for $\mathrm{sign}(\lambda v) < 0$ the limit can be taken, recovering the previous results and the expressions in the two ranges are connected by analytic continuation. It is worth noticing that the energy is a non-polynomial function of N and λ (except for $\xi = 0$), but an effective mass can be introduced as well, in perfect analogy with the zero potential case.

Finally we observe that for all these solutions (for any ξ) the sign of the total momentum P is opposite to that of the coupling constant λ, in spite of the fact that for $\xi > 0$ we can have negative and positive velocity (the total energy has the correct sign in order to realize the described situation). Chirality (in this more general meaning, *i.e.*, $\lambda P < 0$) is therefore a property shared by these scale invariant systems and it is clearly related to the non-linear current interaction present in the relevant Schrödinger equation. At the quantum level we shall recover such a property in the two-body quantum bound state and we shall make some observations for the general case.

B. Solutions around the non-trivial vacuum

We investigate traveling solutions $\phi(x,t)$ that approach, at spatial infinity, a vacuum with density $\rho_0 \neq 0$. When interpreting the standard non-linear Schrödinger equation as a description of a gas of bosons interacting via a repulsive two-body potential, this boundary condition is often called the "condensate boundary condition", since a time-independent solution $\rho = \rho_0$ is usually understood as the Bose condensate. Finite energy solutions with the above boundary are therefore seen as travelling bubbles in a constant background density. On the other hand, the "naive" momentum P is not finite, because of the presence of the "unusual" ρ^4 term in the density $T^0_{\ x}$. In the presence of non-vanishing boundary conditions a conserved charge involves contributions coming from the boundary as well. The correct definition of the total momentum requires the addition of a suitable boundary term, that makes P differentiable (in the functional sense) and therefore compatible with the Hamiltonian structure of the system. The resulting momentum turns out to be finite. This procedure entails a breaking of the relation $E = \frac{v}{2}P$, derived from the scale invariance, a fact that is nevertheless expected because our "condensate boundary condition" explicitly breaks such a symmetry.

We start by presenting the phase of $\hat{\phi}$ when the non trivial boundary condition is taken into account:

$$\theta(x,t) = \frac{mv}{\hbar}\int_{-\infty}^{x-vt}(1-\frac{\rho_0^2}{\rho^2}) - \frac{\lambda}{2}\int_{-\infty}^{x-vt}(\rho^2-\rho_0^2) - \frac{\lambda}{2}\rho_0^2 x + \theta_0 \tag{58}$$

and obviously it approaches to the phase of the vacuum as $x \to \infty$. Then using eq.(58) we obtain the equation for $\rho(y)$

$$\rho'' + \frac{2m}{\hbar}\left(\frac{mv^2}{2\hbar} + \lambda v \rho_0^2\right)\rho - \frac{m^2v^2}{\hbar^2}\frac{\rho_0^4}{\rho^3} - \frac{2mv\lambda}{\hbar}\rho^3 = 0 \tag{59}$$

or equivalently (by using the boundary condition)

$$(\rho')^2 = (\rho^2-\rho_0^2)^2[\frac{mv}{\hbar}\lambda - \frac{m^2v^2}{\hbar^2\rho^2}]. \tag{60}$$

The relevant solution approaching to ρ_0^2 as $y \to \infty$ exists if and only if

$$\lambda v > 0 \qquad \text{and} \qquad \rho^2 > \frac{mv}{\hbar\lambda} \tag{61}$$

The shape of our “dark soliton” is

$$\rho^2(x-vt) = \rho_0^2 \left[1 - \left(1 - \frac{mv}{\lambda\hbar\rho_0^2}\right) \mathrm{sech}^2 \left(\sqrt{\frac{mv\lambda\rho_0^2}{\hbar}\left(1 - \frac{mv}{\hbar\lambda\rho_0^2}\right)}(x-vt)\right)\right]. \qquad (62)$$

The dark soliton is chiral ($\lambda v > 0$) and moving in the opposite direction with respect to the soliton found in the previous section for vanishing cubic potential. Moreover only a finite range of velocity is permitted as clearly displayed by eq.(61): at variance with the soliton case we see that the only free parameter not fixed by the boundary conditions is the velocity v (we keep ρ_0^2 fixed); its absolute value cannot exceed the critical bound $v_{max.} = \frac{\hbar\lambda\rho_0^2}{m}$. The vacuum solution is recovered as v approaches $v_{max.}$.

Let us discuss the conserved quantities: the number, as defined in the previous section, is obviously not finite. It is natural, in the spirit of a bubble interpretation, to call number the quantity

$$N = \int_{-\infty}^{+\infty} dx\,(\rho_0^2 - \rho^2), \qquad (63)$$

that evaluated for eq.(62) is

$$N = \frac{2}{\lambda}\sqrt{\frac{v_{max.}}{v} - 1}. \qquad (64)$$

N is a monotonically decreasing function of v for any choice of background density and coupling constant. Next the expression for the energy is

$$E = \frac{2}{3}\hbar\rho_0^2\sqrt{\frac{v}{v_{max.}}\left(1 - \frac{v}{v_{max.}}\right)^3}. \qquad (65)$$

The naive momentum $P = \int_{-\infty}^{+\infty}\left[mJ + \hbar\frac{\lambda}{2}\rho^4\right]$ is infinite, as we have anticipated, because of the presence of the “non-galilean” term (we recall that $J \to 0$ at infinity). However our Schrödinger equation is a Hamiltonian system with phase space consisting of pairs of functions $(\phi(x), \phi^*(x))$ with boundary condition $(\phi(x)\phi^*(x)) \to \rho_0^2$ and the Poisson bracket

$$\{A, B\} = -i\hbar\int_{-\infty}^{+\infty} d\,x \left(\frac{\delta A}{\delta\phi(x)}\frac{\delta B}{\delta\phi^*(x)} - \frac{\delta A}{\delta\phi^*(x)}\frac{\delta B}{\delta\phi(x)}\right). \qquad (66)$$

To compute the Poisson bracket of P with some other functional we have to compute the functional derivatives $\frac{\delta P}{\delta\phi(x)}$ and $\frac{\delta P}{\delta\phi^*(x)}$: using the expression of P as function the canonical pair we have

$$\delta P = i\frac{\hbar}{2}\rho_0^2 \left(\phi^*\delta\phi - \phi\delta\phi^*\right)\Big|_{-\infty}^{+\infty} + i\hbar\int_{-\infty}^{+\infty} d\,x \left(\delta\phi\partial_x\phi^* - \delta\phi^*\partial_x\phi\right). \tag{67}$$

The functional derivative can be computed only if the first term on the right does not appear, so we have to add a functional that cancels the boundary term: this functional is proportional to θ' and the new momentum can be written in the compact form

$$\hat{P} = \frac{i\hbar}{2}\int_{-\infty}^{+\infty} d\,x \left(\phi^*\partial_x\phi - \phi\partial_x\phi^*\right)\left(1 - \frac{\rho_0^2}{\phi\phi^*}\right). \tag{68}$$

$\hat{P}$ is automatically finite and obviously the subtraction is equivalent to a suitable improvement in the energy-momentum tensor. We define the improved components

$$\hat{T}^0_{\ x} = T^0_{\ x} - \frac{i\hbar\rho_0^2}{2}\partial_x \ln\left(\frac{\phi}{\phi^*}\right), \qquad \hat{T}^x_{\ x} = T^x_x + \frac{i\hbar\rho_0^2}{2}\partial_t \ln\left(\frac{\phi}{\phi^*}\right) \tag{69}$$

automatically satisfying the conservation equation and giving $\hat{P} = \int_{-\infty}^{+\infty} d\,x\hat{T}^0_{\ x}$. We remark that even the integral of $\hat{T}^x_{\ x}$ is convergent. A further observation concerns the scale invariance of the theory in presence of the condensate boundary condition: the generator D (that is not defined using the naive expression for P) is not conserved, as we can expect

$$\frac{dD}{dt} = -\frac{i\hbar\rho_0^2}{2}\int_{-\infty}^{\infty} dx\partial_t \ln\left(\frac{\phi}{\phi^*}\right) \tag{70}$$

The energy-momentum relation is therefore modified as follows

$$H = \frac{v}{2}\hat{P} + \frac{dD}{dt}. \tag{71}$$

The momentum $\hat{P}$ of the dark soliton is easily evaluated

$$\hat{P} = -\text{sign}(\lambda)\hbar\rho_0^2\left[2\text{arctg}\sqrt{\frac{v_{max.}}{v} - 1} + \frac{4}{3}\left(\frac{v}{v_{max.}}\right)\sqrt{\left(\frac{v_{max.}}{v} - 1\right)^3} - \sqrt{\frac{v_{max.}}{v} - 1}\right], \tag{72}$$

that has been checked to be consistent with eq. (71). We notice that $\hat{P}$ is a monotonically decreasing function of v, ranging from $+\infty$ to 0 (in absolute value), while the energy has a maximum value $E = \hbar\frac{\sqrt{3}}{4|\lambda|}\left(\frac{mv_{max.}^2}{2}\right)$ for $v = \frac{v_{max.}}{4}$. It is interesting to obtain the dispersion relation for these solutions, namely $\frac{dE}{d\hat{P}}$. In a particle-like solution we expect $\frac{dE}{d\hat{P}} = v$. In ref. [16] the same subtraction procedure was followed in a Galilean invariant theory for dark soliton solutions: in that case the particle-like dispersion relation was recovered. This is

not true in our case: performing the variation of the energy respect $\hat{\phi}^*$ and $\hat{\phi}$ and using the equation of motion we get

$$\delta E = v\delta \hat{P} - v\frac{\hbar\lambda}{2}\rho_0^2 \delta N, \tag{73}$$

and therefore

$$\frac{dE}{dv} = v\frac{d\hat{P}}{dv} - v\frac{\hbar\lambda}{2}\rho_0^2\frac{dN}{dv}. \tag{74}$$

Because the theory is not Galilean invariant the number depends on v, as we have seen, once ρ_0^2 is considered a fixed parameter: the explicit computation gives

$$\frac{dE}{d\hat{P}} = 2v\frac{(v_{max.} - v)(v_{max.} - 4v)}{8v^2 - 10vv_{max.} - v_{max.}^2}, \tag{75}$$

which does not resemble the particle-like behavior. The curve $E(\hat{P})$ can be easily plotted

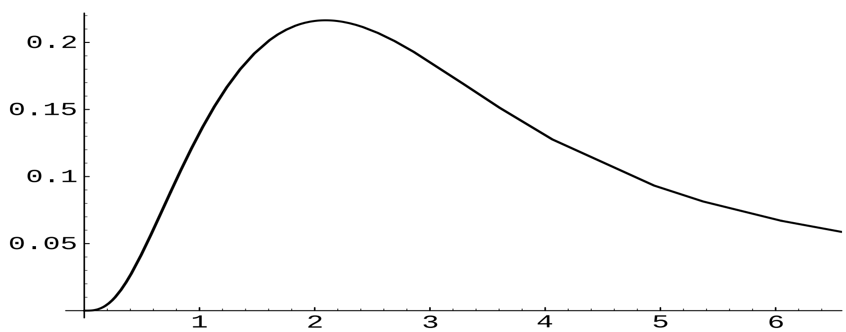


FIG. 1. The x-axis is $-\mathrm{sign}(\lambda)\ \dfrac{P}{\hbar\rho_0^2}$, the y-axis is $\dfrac{E}{\hbar\rho_0^2}$.

The total energy goes to zero for small and large momenta and in particular as $\hat{P} \to 0$ we have

$$E \simeq \frac{2}{3}\frac{|\lambda|}{\hbar m \rho_0^2}|\hat{P}|^3. \tag{76}$$

We have seen that the finite-energy condition is equivalent to the requirement that $J(\pm\infty) = 0$. We can also imagine physical situations in which the asymptotic current is not zero; in this case a subtraction procedure can derived from the differentiability requirement, leading

to a sensible definition of the total energy. The general structure of the dark solitons in this case will be reported elsewhere.

We turn now, as in the previous subsection, to the problem of generalizing the above analysis when a cubic potential in ρ^2 is added. A finite energy configuration requires that asymptotically $\hat{\phi}$ approaches a static solution with constant modulus and zero energy: the phase of these waves must satisfy

$$(\theta' + \frac{\lambda}{2}\rho_0^2)^2 = \frac{\lambda^2}{4}\xi\rho_0^4. \tag{77}$$

Therefore only for $\xi \geq 0$ can we obtain the static, zero-energy solution

$$\theta = -\frac{\lambda}{2}\rho_0^2\left(1 \mp \sqrt{\xi}\right)x + \theta_0 \tag{78}$$

that carries the asymptotic current $J = \pm\frac{\hbar\lambda}{2m}\sqrt{\xi}\rho_0^4$. When considering dark solitons interpolating these solutions at $\pm\infty$, this asymptotics reflects the following choice for J_∞:

$$J_\infty(t) = -v\rho_0^2 \pm \frac{\hbar\lambda}{2m}\sqrt{\xi}\rho_0^4. \tag{79}$$

The relevant equation for the phase is easily solved

$$\theta = \left(\frac{mv}{\hbar} \mp \frac{\lambda}{2}\sqrt{\xi}\rho_0^2\right)\int_{-\infty}^{x-vt} d\,y\left(1-\frac{\rho_0^2}{\rho^2}\right) - \frac{\lambda}{2}\int_{-\infty}^{x-vt} d\,y(\rho^2-\rho_0^2) - \frac{\lambda}{2}\rho_0^2\left(1\mp\sqrt{\xi}\right)x + \theta_0(t), \tag{80}$$

while for the modulus we get

$$(\rho')^2 = (\rho^2-\rho_0^2)^2\left[\frac{mv}{\hbar}\lambda - \left(\frac{mv}{\hbar} \mp \frac{\lambda\sqrt{\xi}}{2}\rho_0^2\right)^2\frac{1}{\rho^2} - \frac{\lambda^2}{4}\xi(\rho^2+2\rho_0^2)\right] \tag{81}$$

From the analysis of the above equation we can derive the parameter range for which solitonic solution exist: one has to solve a system of inequalities and here we state only the result, deferring details and physical interpretation to a forthcoming paper focused on "condensate boundary condition" solutions.

a.) The existing solutions are chiral with $\mathrm{sign}(\lambda v) > 0$;

b.) There is an intrinsic lowest velocity

$$|v| > \frac{4}{3}\frac{\hbar|\lambda|}{m}\xi\rho_0^2$$

There is no solution with finite energy unless $0 \le \xi \le 1$;

In this range we have two different kinds of solutions: setting

$$
\begin{aligned}
A &= -\frac{\beta v}{4\alpha^2} - \frac{3\beta^2\xi\rho_0^2}{16\alpha^2} - \frac{(2v \mp \sqrt{\beta^2\xi\rho_0^2})^2}{16\alpha^2\rho_0^2}, \\
B &= \frac{\beta v}{4\alpha^2} + \frac{\beta^2\xi\rho_0^2}{4\alpha^2}, \\
C &= -\frac{\beta^2\xi\rho_0^2}{16\alpha^2}, \\
\Delta &= B^2 - 4AC,
\end{aligned}
\tag{82}
$$

with $\alpha = \frac{\hbar}{2m}$ and $\beta = -\frac{\hbar\lambda}{2m}$ we find a true dark soliton

$$
\rho^2 = \rho_0^2\left(1 - \frac{2A}{\sqrt{\Delta}\cosh(2\rho_0\sqrt{A}(x-vt)) - B}\right), \tag{83}
$$

that represents an "hole" in the background density and a "bright" soliton

$$
\rho^2 = \rho_0^2\left(1 + \frac{2A}{\sqrt{\Delta}\cosh(2\rho_0\sqrt{A}(x-vt)) + B}\right), \tag{84}
$$

that is a positive density excitation over the background. Obviously while the first one goes smoothly to the solution for $\xi = 0$ the second one has a singular limit. The upper limit on the velocity is determined by the positivity of Δ.

We end this section by remarking that "condensate boundary condition" solutions have no clear interpretation in the quantum theory; although their quantum role is an open problem they find applications in such branches of physics as condensed matter and plasma dynamics.

IV. QUANTUM THEORY

The simplest non relativistic example of the relation between solitons and quantum bound states is represented by the usual nonlinear Schrödinger equation. One may view eq. (9) as a Heisenberg equation for the quantum field $\hat{\phi}(x,t)$, which is taken to satisfy the commutation relation

$$[\hat{\phi}(x_1,t),\hat{\phi}^*(x_2,t)] = \delta(x_1 - x_2). \tag{85}$$

The Hilbert space can be decomposed into invariant subspaces according to the integer eigenvalues n of the (conserved) number operator $\int_{-\infty}^{\infty} dx\, \hat{\phi}^* \hat{\phi}$ and the n-body wave function,

$$\phi_n(x_1,...,x_n,t) = \frac{1}{\sqrt{n}} < 0|\hat{\phi}(x_1,t)...\hat{\phi}(x_n,t)|n >, \tag{86}$$

satisfies a Schrödinger equation with two-body, pairwise attractive δ-function interactions. The bound-state spectrum can be explicitly computed [18] and in the center of mass frame it is given by

$$E_n = -\frac{\lambda^2 m}{6\hbar^2}(n^3 - n). \tag{87}$$

On the other hand, the semiclassical quantization of the one soliton solution at the leading order produces the identification of the classical $U(1)$ charge $N = \int_{-\infty}^{+\infty} dx\ \hat{\phi}^*_{soliton}\hat{\phi}_{soliton}$ with the quantum number n and the spectrum

$$E_{Semicl.}(n) = E_{Class.}(n) = -\frac{\lambda^2 m}{6\hbar^2} n^3. \tag{88}$$

Note that the classical energy of the soliton $E_{Class.}$ gives the correct leading term as $\lambda \to 0$ and λn is kept fixed. Remarkably the next-to-leading correction diminishes n^3 by n ($n^3 \to n^3 - n$) and it is exact. From the quantum field theory point of view the correction linear in n to the energy stems from a renormalization effect, namely from the counterterm $m(\Lambda)\hat{\phi}^*\hat{\phi}$ necessary to cancel the divergences of the theory.

One can argue on general ground [2] and verify by computation [1,12,17] that all integrable systems enjoy a similar stringent analogy between quantum bound states and solitons; much less is known for nonintegrable ones[1]. Our model ($\xi = 0$) appears to belong to this second category: indeed it fails to pass the Painlevé test. Nevertheless it can be considered as a perturbed version of the derivative nonlinear Schrödinger equation ($\xi = 1$) [20], for

[1]From the point of view of the form factor approach, the question was recently investigated in [19], where a a non-integrable perturbation of the Ising model in external magnetic field was considered.

which some evidences of quantum integrability [21] exist. Thus it seems reasonable to expect that the quantization of the corresponding one-soliton solution will provide, in this case, a good approximation of the quantum spectrum. However before committing ourselves with the semiclassical approach, we shall try to extract some information through the pattern described at the beginning of this section.

We take for the quantum hamiltonian the normal ordered expression

$$H = \frac{\hbar^2}{2m} \int dx : \left| \left(\partial_x + i \frac{\lambda}{2} \rho^2 \right) \phi \right|^2 : \tag{89}$$

and we posit the canonical commutation relations (85). The normal ordering prescription has been adopted to define properly the quantum Hamiltonian. This corresponds to removing all the singular interactions proportional to $\delta(0)$ in the resulting n-body Schrödinger equations. The above procedure is not without price, entailing in fact the loss of positivity for the quantum energy. [The classical Hamiltonian <u>is</u> positive definite for *any* value of the coupling constant λ.] With this choice, the n-body Schrödinger equation becomes

$$-\frac{\hbar^2}{2m} \left[\sum_{i=1}^{n} \partial^2_{x_i} + 2i\lambda \sum_{i<j}^{n} \delta(x_i - x_j) \partial_{x_i} - \frac{\lambda^2}{4} \sum_{i \neq j \neq k}^{n} \delta(x_i - x_j)\delta(x_i - x_k) \right] \phi_n = E_n \phi_n. \tag{90}$$

In eq. (90) the quantum non-integrability manifests itself through the presence of a three-body interaction generated by the $(\phi^*\phi)^3$ part of the Hamiltonian. In the case of the DNLS equation ($\xi = 1$) this term is, instead, cancelled by an analogous contribution coming from the potential V. By translational invariance we can always separate the center of mass motion by introducing the parametrization

$$\phi_n(x_1, ..x_n) = \exp\left[i \sum_{i=1}^{n} \frac{P x_i}{n} \right] \chi_n(x_1, ..x_n), \qquad \text{with} \qquad \sum_{i=1}^{n} \partial_{x_i} \chi_n(x_1, ..x_n) = 0, \tag{91}$$

namely χ_n depends only on the relative coordinates. However, because of the lack Galilean invariance, the total momentum P will not decouple completely and will appear as a parameter in the reduced Schrödinger equation for χ_n.

From (90) we note that the claim in [9] of statistical transmutation for the quantum excitation of this model is inexact. It is impossible to remove the δ interaction by a phase redefinition and therefore no change in the statistical behaviour should be expected [10].

A. The two-body problem

We begin our quantum investigations by considering the simplest sector contained in the Hilbert space of the field theory defined by the quantum Hamiltonian (89): the two-particle one. Using eq. (91) and defining $x \equiv x_1 - x_2$ we find that χ_2 satisfies

$$\left(-\frac{\hbar^2}{m}\frac{\partial^2}{\partial x^2} + \frac{P^2}{4m} + \hbar\frac{\lambda P}{2m}\delta(x)\right)\chi_2(x) = E_2\chi_2(x). \tag{92}$$

The presence of the total momentum P in the $\delta-$function potential for the relative motion vividly demonstrates the absence of Galilean invariance. Provided that

$$\lambda P < 0 \tag{93}$$

eq. (92) possesses a bound state solution with energy

$$E_2 = \frac{P^2}{4m}\left(1 - \frac{\lambda^2}{4}\right) = \frac{P^2}{2M_{B.S.}}. \tag{94}$$

Since $P/2m$ can be identified with the velocity v of the center of mass, we recognize the condition (93) as the one (see eq. (44)) that guarantees the existence of the soliton (45) at the classical level. Remarkably even the scale invariance is preserved: in fact the dispersion relation (94) is unchanged under dilatation. [Unfortunately this property will fail in the n-body problem.] On the other hand the energy E_2 and therefore the effective mass of the bound state

$$M_{B.S.} = \frac{2m}{1 - \lambda^2/4} \tag{95}$$

become negative for $\lambda^2 > 4$. The loss of positivity originates from the normal ordering adopted in defining the quantum Hamiltonian. As a three-body interaction of the form $(\phi^*\phi)^3$ can never contribute to the two body problem, the bound state (94) is, actually, shared by all the family (20) of models defined in sect. 2 and parametrized by the coupling constant ξ. Moreover all the solitons in this class of models enjoy the property $\lambda P < 0$ in perfect analogy with eq. (93).

It is instructive to recover the same results in a field theory framework, namely by resumming the perturbative series defined in terms of Feynman graphs. This approach will

be most useful in the discussion of the $n-$body case. The quantum propagator for ϕ is easily derived

$$D(x,t) = \frac{1}{(2\pi)^2} \int dk\ d\omega\ \frac{e^{-i(\omega t - kx)} i}{\omega - \frac{k^2}{2m} + i\epsilon}, \tag{96}$$

($\hbar$ is put equal 1 from now on) and the interaction vertices are

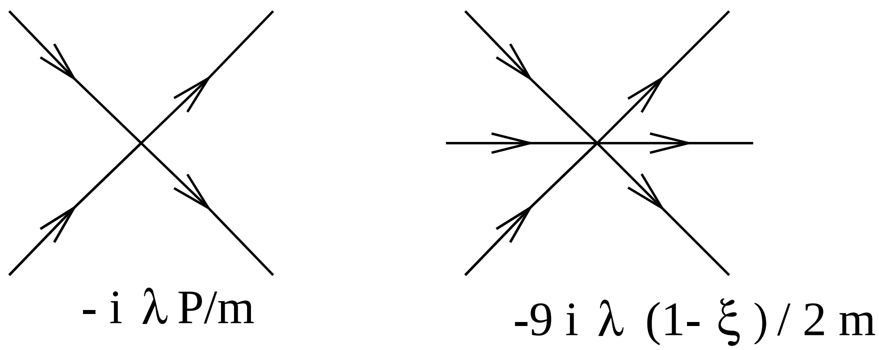


FIG. 2. Feynman rules: P is the total incoming momentum.

At one-loop level the relevant graph is

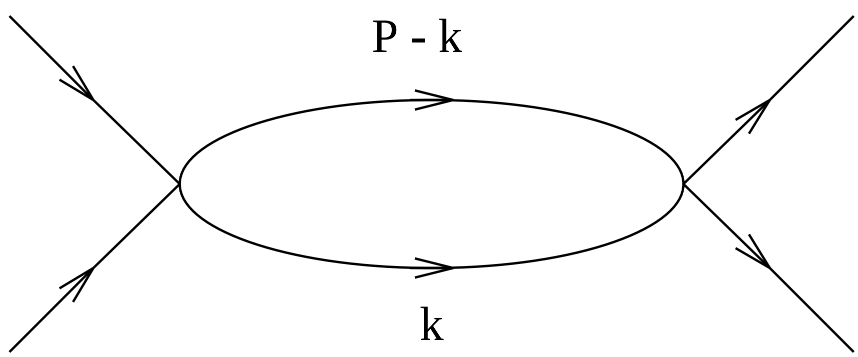


FIG. 3. One loop contribution to the four point vertex. P is the total incoming momentum, k the loop momentum.

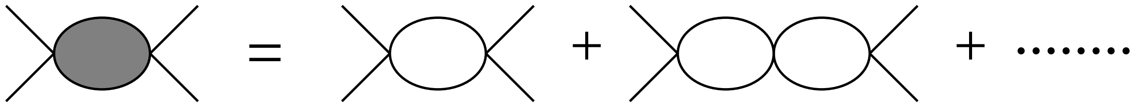
The absence of the t and u channel implied by the non-relativistic character of the theory and moreover the absence of a tadpole contribution, because of the normal-ordering, greatly simplifies the final result. Choosing [2] $P^2 > 4mE$, we get the one-loop amplitude

$$A_1(P,E,\lambda) = -i\frac{\lambda^2 P^2}{2m} \frac{1}{\sqrt{P^2 - 4mE}}. \tag{97}$$

---

[2]This restriction is the equivalent of the condition $\gamma > 0$ found in the discussion of the classical soliton in subsect. 3A

Here $P = p_1 + p_2$ is total momentum and $E = \omega_1 + \omega_2$ is the total energy. To obtain the full amplitude we have only to resum the geometric series



because there is no contribution from higher loops or from the six-vertex. [Recall in fact that the number of particles must be conserved in each stage of our diagrams since we are in a non-relativistic theory.] Thus we obtain

$$A(P, E, \lambda) = i\frac{\lambda P}{m} \frac{\sqrt{P^2 - 4mE}}{\lambda P/2 + \sqrt{P^2 - 4mE}}. \tag{98}$$

We have a pole for $\lambda P < 0$ at the energy $E = \frac{P^2}{4m}\left(1 - \frac{\lambda^2}{4}\right)$, recovering therefore the results of the two-body Schrödinger equation. We remark that no infinity arises in the perturbative evaluation of the two-body scattering amplitude (once the normal-ordering is adopted) and no renormalization scale is needed: scale-invariance is maintained as the classical energy-momentum relation clearly shows.

B. The n-body problem

For the general n-body problem we did not succeed in computing the energy eigenvalues for the bound states exactly because of the presence of the three-body interaction that complicates the eigenvalue equation. Nevertheless starting from the usual non-linear Schrödinger equation we can construct a perturbative solution to the problem, that is reasonable for certain values of the parameters and is in agreement with the soliton characteristics. The reduced equation is

$$-\frac{1}{2m}\left[\sum_{i=1}^{n} \partial_{x_i}^2 - \frac{2\lambda P}{n}\sum_{i<j}^{n} \delta(x_i - x_j) + 2i\lambda \sum_{i<j}^{n} \delta(x_i - x_j)\partial_{x_i} - \right.$$
$$\left. \frac{\lambda^2}{4}\sum_{i\neq j\neq k}^{n} \delta(x_i - x_j)\delta(x_i - x_k)\right]\chi_n = \left(E_n - \frac{P^2}{2mn}\right)\chi_n. \tag{99}$$

Defining $z_i = -\frac{2\lambda P}{n} x_i$ we obtain

$$\left[\underbrace{\sum_{i=1}^{n} -\partial_{z_i}^2 + \text{sign}(\lambda P) \sum_{i<j}^{n} \delta(z_i - z_j)}_{\mathcal{H}_0} - \lambda \underbrace{i\text{sign}(\lambda P) \sum_{i<j}^{n} \delta(z_i - z_j) \partial_{z_i}}_{\mathcal{H}_1} + \right.$$

$$\left. \frac{\lambda^2}{4} \underbrace{\sum_{i \neq j \neq k}^{n} \delta(z_i - z_j)\delta(z_i - z_k)}_{\mathcal{H}_2} \right] \chi_n = \frac{mn^2}{2P^2\lambda^2} \left(E_n - \frac{P^2}{2mn} \right) \chi_n. \tag{100}$$

We recognize that the eigenvalue equation can be rewritten as the NLS one plus two perturbations proportional to λ and λ^2

$$\left[\mathcal{H}_0 - \lambda \mathcal{H}_1 + \frac{\lambda^2}{4} \mathcal{H}_2 \right] \chi = \mathcal{E}_n \chi. \tag{101}$$

The NLS equation defined by $\mathcal{H}_0$ admits bound states only if $\lambda P < 0$ in agreement with the chiral nature of the soliton; thus we shall consider only this case. The zero-order problem $\mathcal{H}_0 \chi_n^{(0)} = \mathcal{E}_n^{(0)} \chi_n^{(0)}$ gives

$$\mathcal{E}_n^{(0)} = -\frac{1}{48}(n^3 - n) \tag{102a}$$

$$\chi_n^{(0)} = \sqrt{\frac{[2(n-1)]!!}{4^{n-2} n}} \exp\left[-\frac{1}{4} \sum_{i \neq j} |z_i - z_j| \right], \tag{102b}$$

and therefore the "zero-order" energy

$$E_n^{(0)} = \frac{P^2}{2mn} \left(1 - \frac{n^2 - 1}{12} \lambda^2 \right). \tag{103}$$

This result is correct as $\lambda \to 0$, $\dfrac{P\lambda}{n}$ is kept constant. Moreover the first correction, coming from $\mathcal{H}_1$ vanishes. Infact if $\mathcal{E}_n = \mathcal{E}_n^{(0)} + \lambda \mathcal{E}_n^{(1)} + ..$ the first order correction is simply

$$\mathcal{E}_n^{(1)} \propto \int_{\infty}^{\infty} \Pi_{j=1}^{n} dz_j (\chi_n^{(0)})^* (\sum_{i<j}^{n} \delta(z_i - z_j) \partial_{z_i}) \chi_n^{(0)} \propto \int_{\infty}^{\infty} \Pi_{j=1}^{n} dz_j (\chi_n^{(0)})^* \chi_n^{(0)} (\sum_{i \neq k}^{n} \text{sign}(z_i - z_k)) = 0, \tag{104}$$

because $(\sum_{i \neq k}^{n} \text{sign}(z_i - z_k)) = 0$. Therefore

$$E_n = \frac{P^2}{2mn} \left(1 - \frac{n^2 - 1}{12n} \lambda^2 \right) + O(\lambda^4). \tag{105}$$

Obviously for $n = 2$ the result is exact: for generic n it shows that in a suitable limit the equation is essentially equivalent to NLS equation with effective coupling constant $\frac{\lambda P}{n}$. It is interesting to confront the perturbative solution with the exact energy spectrum of DNLS equation ($\mathcal{H}_2 = 0$), which has been recently computed [20]

$$E_n^{DNLS} = \frac{P^2}{4m} \frac{\lambda}{\tan\left(n \arctan \frac{\lambda}{2}\right)} \simeq \frac{P^2}{2mn}\left(1 - \frac{n^2-1}{12n}\lambda^2\right) + O(\lambda^4). \tag{106}$$

The agreement between the perturbative and the exact result provides a strong check of our approach.

C. Trace anomaly

In subsec. 4A we have seen how to recover the two-body states by means of Feynman graphes. Here we would like to take advantage of the field theory approach to derive another exact result (connected to scale invariance) that would otherwise be quite difficult to establish. Classically the theory is invariant under dilatation; at the quantum level, instead, the ultraviolet divergences we encounter in perturbatively evaluating scattering amplitudes will spoil this property. In fact the well-known machinery of renormalization allows us to remove infinities consistently, but the price to be paid is the introduction of a "renormalization scale" that obviously breaks scale symmetry. In quantum mechanical language it is the highly singular three-body interaction that needs to be defined: different choices correspond to different self-adjoint extensions of a Schrödinger operator associate with the $n-$ body problem. The (dimensional) parameter describing the possible extensions is usually related to the renormalized coupling constant [22].

A power counting analysis of the relevant diagrams shows that all divergences arise from the two loop six-point function or from its iterations (see figure below). No infinity comes from the 4-point vertex.

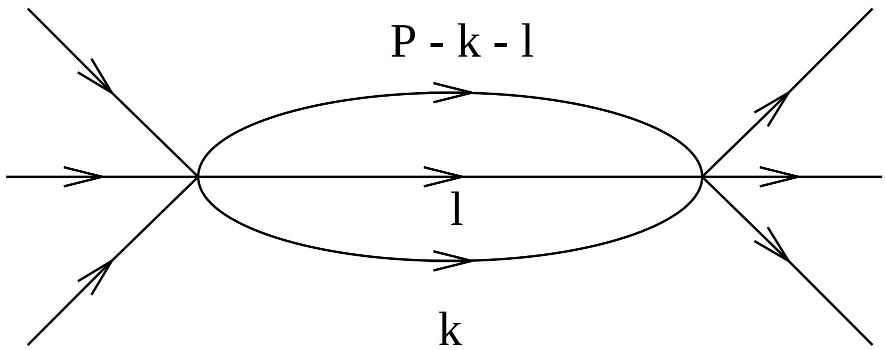


FIG. 4. Divergent 2-loop contribution to the six vertex function. P is the total incoming momentum; k and l are the loop momenta.

The two-loop amplitude in fig. 4. can be easily written down with the help of the Feynman rules given in sect. 4A. After integrating over the energy ω_1 and ω_2 in the loops we are left with the following integral over the momenta $\left(\alpha_0 = -\frac{9}{2m}\lambda^2(1-\xi_0)\right)$

$$
\begin{aligned}
A_6(E,P) &= -\frac{i\alpha_0^2}{24\pi^2}\int_{-\infty}^{+\infty} dl dk \, \frac{1}{E - \frac{l^2}{2m} - \frac{k^2}{2m} - \frac{(P-l-k)^2}{2m} + i\epsilon} \\
&= \frac{i\alpha_0^2 m}{12\sqrt{3}\pi^2}\int_{-\infty}^{+\infty} dk \; dl \; \frac{1}{l^2 + k^2 + \left(\frac{P^2}{6} - mE\right) - i\epsilon}.
\end{aligned} \tag{107}
$$

Here, as before, E and P stand for the total energy and momentum respectively. [Although we are mainly interested in $\xi = 0$ case, we are forced to introduce a bare coupling constant $\xi_0 \neq 0$ in order to have a renormalizable theory.] The final integral in eq. (107) is logarithmically divergent. Upon introducing a cutoff Λ and a scale μ, we can identify the singular contribution

$$
A_6^{div.}(E,P) = \frac{i\alpha_0^2 m}{12\sqrt{3}\pi} \log\frac{\Lambda^2}{\mu^2}. \tag{108}
$$

The divergent term can be reabsorbed in a redefinition of the coupling constant ξ by defining a ξ_R as follows

$$
\xi_0 = \xi_R - \frac{\sqrt{3}}{8\pi}\lambda^2(1-\xi_R)^2 \log\frac{\Lambda^2}{\mu^2}. \tag{109}
$$

Since all the divergencies come from iterating the elementary graph in fig. 4 (recall, in fact, that insertions of the four-point vertex do not give additional contributions to the divergent part), the result can be therefore made exact

$$1 - \xi_0 = \frac{1 - \xi_R}{1 + \frac{\sqrt{3}}{8\pi}\lambda^2(1 - \xi_R)\log\frac{\Lambda^2}{\mu^2}}. \tag{110}$$

The β-function obtained from this relation is

$$\beta(\xi) = \mu\frac{\partial}{\partial\mu}\xi_R(\mu) = -\frac{\sqrt{3}}{4\pi}\lambda^2(1 - \xi_R)^2. \tag{111}$$

We remark that this result is exact and some of its consequences can be discussed. Firstly the coefficient of the three-body interaction appears to be running (it depends on the scale μ^2): in field theoretical language it means that at the quantum level a potential $V(\rho^2) \simeq \rho^6$ is induced from loop corrections. We are therefore naturally led to consider the general family parametrized by ξ. Secondly the fixed point of $\beta(\xi)$ is for $\xi_R = 1$, the DNLS equation, that is the integrable system: scale invariance is maintained in this model at quantum level as the exact expression, *e.g.*, for the bound-state energy displays:

$$E = \frac{P^2}{2M}, \qquad M = m\frac{\tan(N \arctan\lambda)}{\lambda}. \tag{112}$$

On the other hand for $\xi_R \neq 1$ we expect that the classical energy-momentum relation is broken at the quantum level. To make this idea more precise, we must relate the scale dependence of the quantum theory to the non conservation of the generator D of the dilatation. This can be done easily along the line of [23] where the same problem was discussed in 2+1 dimension for a ρ^4 theory: the methods relies in deriving a set of Ward identities from scale invariance. Following [23] we obtain for the proper vertices in momentum-space $\Gamma^n(p_i, \omega_i, \xi_R, \mu)$ the Ward identity

$$[\partial_a + (3 - \frac{n}{2})]\Gamma^n(e^{-a}p_i, e^{-2a}\omega_i, \xi_R, \mu) = -i\Gamma^n_T(0, e^{-a}p_i, e^{-2a}\omega_i, \xi_R, \mu), \tag{113}$$

$\Gamma^n_T(0, e^{-a}p_i, e^{-2a}\omega_i, \xi_R, \mu)$ being the proper vertex derived from the Green function

$$G^n_T(y, x_1, ..x_n) =< 0|T[(2T^{00}(y) - T^{xx}(y))\phi(x_1)...\phi(x_n)]|0 >, \tag{114}$$

while a is the dilatation parameter. If scale-invariance holds at the quantum level, the right-hand side of eq.(113) is zero. On the other hand the renormalization group equation, derived in the usual way is (we recall that no wave-function renormalization is present)

$$[\frac{\partial}{\partial a} + \beta(\xi_R)\frac{\partial}{\partial \xi_R} + (3-\frac{n}{2})]\Gamma^n(e^{-a}p_i, e^{-2a}\omega_i, \xi_R, \mu) = 0. \tag{115}$$

Comparing eq.(113) and eq.(115) we see that scale invariance is broken by the $\beta(\xi_R)$ term. More precisely we can write the anomalous Ward identity

$$\beta(\xi_R)\frac{\partial}{\partial \xi_R}\Gamma^n = i\Gamma^n_T, \tag{116}$$

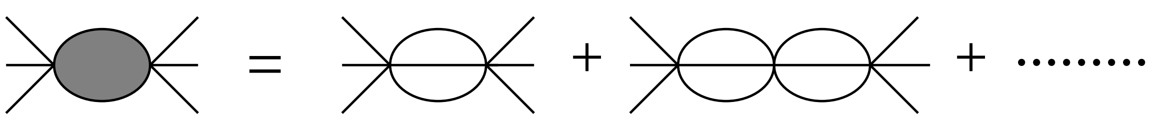
that reflects in a quantum equation for the trace of the energy-momentum tensor

$$T^{xx} = 2T^{00} + \frac{\sqrt{3}}{32\pi}\frac{\lambda^4}{m}(1-\xi_R)^2(\phi^*\phi)^3. \tag{117}$$

The scale anomaly is

$$\frac{dD}{dt} = -\frac{\sqrt{3}}{64\pi}\frac{\lambda^4}{m}(1-\xi_R)^2\int_{-\infty}^{+\infty}(\phi^*\phi)^3. \tag{118}$$

From the above analysis it appears clear that physical quantities, like the bound-states energies E_n, must depend both on $\xi_R(\mu)$ and μ so that $\mu\frac{\partial}{\partial\mu}E_n(\xi,\mu) = 0$. For example it is not difficult to compute the three-body bound-state in a simpler model, with potential purely proportional to ρ^6 ($V(\rho^2) = -\frac{\alpha_0}{(3!)^2}\rho^6$; the family of models considered here reduces to this one in the limit $\lambda \to 0$ but $\lambda^2\xi_0 = 2m\alpha_0/9$ is kept fixed). In this case, due to the absence of the two-body interaction, we have only to resum the geometric series



leading to the exact scattering amplitude

$$A(P,E,\alpha) = \frac{i\alpha_R}{m}\frac{1}{1 - \frac{\alpha_R}{12\sqrt{3}\pi}\log\frac{\mu^2}{P^2 - 6mE}} \tag{119}$$

that exhibits a pole for

$$E_3 = \frac{P^2}{6m} - \frac{\mu^2}{6m}\exp[-\frac{12\sqrt{3}\pi}{\alpha_R(\mu^2)}]. \tag{120}$$

The second term, that is actually independent from μ^2 because the dependence of α_R on μ^2 is so chosen that this happens, is generated by the scale anomaly and it breaks the classical energy-momentum relation. In term of the physical parameters the scattering amplitude is

$$A(P,E,E_3) = \frac{12\sqrt{3}\pi i}{m\,\log\left[\dfrac{P^2-6E_3m}{P^2-6Em}\right]}, \tag{121}$$

that depends on one more dimensionful parameter (E_3). Moreover we see that a finite energy bound-state requires the bare coupling constant α_0 to be positive, implying that only the attractive potential leads to non-trivial physics, in close analogy with the analysis performed in [23]. In this particular case the non-trivial interaction can also be viewed as a self-adjoint extension of the two-dimensional laplacian, with one point removed. The self-adjoint extension parameter has the interpretation of the renormalized coupling constant α.

In our case the resummation, even in the simplest case of the three-body scattering amplitude, is not a geometric series: it is not clear if only a bare parameter $\xi_0 > 1$ leads to a bound-state or if the derivative coupling (for $\lambda P < 0$) can bind particles even in presence of a bare repulsive three-body potential. In the next subsection we will try to combine the soliton structure of the theory with the trace anomaly in order to get some information on the quantum bound-state of the theory.

D. Solitons and bound states

The semiclassical quantization of the one-soliton solution needs essentially the computation of the small fluctuation around the soliton: following the approach of DHN [1], we have to calculate a functional determinant that produces the first quantum correction to the classical energy. In integrable systems, like NLS, DNLS or Sine-Gordon, one can use the multi-soliton solutions to compute the stability angles [1], from which the determinant can be determined. In particular, in all these systems, the contribution of the determinant consists in a (finite) renormalization of the parameters appearing in the initial lagrangian. In the DLNS case, *e.g.*, the soliton has the mass $M_{Cl.}(N) = \dfrac{m\tan N\lambda}{\lambda}$ while the mass of the quantum $n-$body bound-state is $M_{Q.}(n) = \dfrac{m\tan[n\arctan\lambda]}{\arctan\lambda}$ [20]. The relevant renormalization is derived from the ratios

$$\frac{M_{Cl.}(N)}{M_{Cl.}(1)} = \frac{\tan N\lambda}{\tan\lambda}, \qquad \text{and} \qquad \frac{M_{Q.}(n)}{M_{Q.}(n)} = \frac{\tan[n \arctan\lambda]}{\tan[\arctan\lambda]}, \tag{122}$$

therefore consisting in the replacement $\lambda \to \arctan\lambda$. [In general we must expect that only the mass ratios are correctly given by the semiclassical approximation.] In Sine-Gordon we have a similar renormalization for the coupling constant β^2

$$\beta^2 \to \frac{\beta^2}{1 - \beta^2/8\pi}.$$

The identification of the number parameter N, with the quantum number n, representing the number of particles bounded can be obtained applying the Bohr-Sommerfield rule

$$dE = \omega dn, \tag{123}$$

to the classical energy and taking ω the phase frequency at the maximum of the moving soliton (for $x - vt = 0$).

In our case we do not have the two-soliton solution to use for the computation of the stability angles and it is not clear what is the structure of the subleading quantum corrections. We can nevertheless compute the energy using the Bohr-Sommerfield formula, neglecting the contribution of the functional determinant. This can be easily done for the general family of scale invariant solitons, parametrized by ξ: we take the energies of the classical soliton (55) and (56) and we use the phase eq.(40), to obtain the frequency ω that turns out to be

$$\omega = \omega_0 + \frac{mv^2}{\hbar}. \tag{124}$$

We rewrite the Bohr-Sommerfield formula as

$$\frac{dE}{dN} = \omega \frac{dn}{dN}, \tag{125}$$

that can be integrated to give $n = N$. The "zero" order energy for the n-body bound state is therefore

$$E^{(0)}_{Q.}(n) = E_{Cl.}(n).$$

Let us first study the case $\xi = 0$; we expect that this equivalence is correct for $\lambda \to 0$. In fact

$$\frac{E^{(0)}_{Q.}(n)}{E^{(0)}_{Q.}(1)} = \frac{1+\dfrac{\lambda^2}{12}}{1+\dfrac{n^2\lambda^2}{12}} = \frac{1}{n}(1-\frac{\lambda^2}{12}(n^2-1)) + O(\lambda^4). \tag{126}$$

We stress that this relation is valid to $O(\lambda^4)$ but in the case of $n = 2$ it is exact (the quantum corrections must compensate the higher terms in the λ-expansion). In the general case we see that eq.(126) is in agreement with the perturbative expansion developed in Subsect.3. However we have seen that from renormalization the general scale-invariant potential is induced and we are naturally led to the ξ-family of theories. We find that for $\xi > 0$ different answers are given, in the small-λ limit, depending on the soliton chosen:

$$\begin{aligned}\frac{E^{(0)}_{Q.}(n)}{E^{(0)}_{Q.}(1)} &= \frac{1}{n}(1-\frac{\lambda^2}{12}(n^2-1)) + O(\lambda^4), \quad \lambda v < 0,\\ &= \frac{1}{n}(1+\frac{\lambda^2}{12}\frac{\xi}{\xi-2}(n^2-1)) + O(\lambda^4), \quad \lambda v > 0.\end{aligned} \tag{127}$$

Only for $\xi = 1$ do the two results coincide, while in general only for $\lambda v < 0$ do we recover the "perturbative" computation: this is easily understood by realizing that only the soliton with $\lambda v < 0$ is connected to an attractive NLS (that is the zero-order of the perturbative expansion). The meaning of the soliton with $\lambda v > 0$ is not clear; moreover the entire procedure, except that for $\xi = 1$, can be trusted only for $P^2 \to \infty$ and $\lambda \to 0$, because we are neglecting the trace anomaly that drastically changes the momentum-energy relation respected by our "zero"-order approximation. Only in the large momentum limit does the classical term dominate over the "anomalous" contribution, as can be inferred from the analysis of the pure ρ^6 theory.

In the following, in order to improve our knowledge about the quantum spectrum, we shall try to understand how the trace anomaly may modify the soliton analysis. We shall start assuming that the eigenvalue E_n has the form

$$E_n = \frac{\mu^2}{m} F\left(\lambda, \xi_R(\mu^2), n, \frac{P^2}{\mu^2}\right) \tag{128}$$

The particular dependence on the mass m can be inferred from the $n-$body Schrodinger equation. The factor μ^2 has been pulled out to make the function F dimensionless. Since

the energy levels are physical observables, they must be independent of the subtraction point μ, *i. e.* $\mu\frac{\partial}{\partial\mu}E_n(\mu)=0$, we have

$$t\frac{\partial F}{\partial t}-\frac{\beta(\xi)}{2}\frac{\partial F}{\partial \xi}-F=0, \tag{129}$$

where t stands for $\frac{P^2}{\mu^2}$. It is not difficult to verify that the general solution for this equation is

$$F=\frac{P^2}{\mu^2\mathcal{M}\left[\lambda,n,\frac{\mu^2}{P^2}\exp\left(\frac{8\pi}{\sqrt{3}\lambda^2(1-\xi(\mu))}\right)\right]} \tag{130}$$

or equivalently

$$E_n=\frac{P^2}{m\mathcal{M}\left[\lambda,n,\frac{\mu^2}{P^2}\exp\left(\frac{8\pi}{\sqrt{3}\lambda^2(1-\xi(\mu))}\right)\right]} \tag{131}$$

The new dependence on the combination $\mu^2\exp\left(\frac{8\pi}{\sqrt{3}\lambda^2(1-\xi(\mu))}\right)$ is just the symptom of the appearance of a new scale in the theory. This, for example, may be identified with the energy of the three-body bound state and it is obviously renormalization group invariant.

In the limit $P^2\to\infty$, we expect a resurrection of scale invariance and consequently eq. (131) must match, in this limit, the soliton analysis. Taylor-expanding (131) we get

$$E_n=\frac{P^2}{m\mathcal{M}_0\left[\lambda,n\right]}+\mathcal{M}_1[n,\lambda]\mu^2\exp\left(\frac{8\pi}{\sqrt{3}\lambda^2(1-\xi(\mu))}\right)+O\left(\frac{1}{P^2}\right). \tag{132}$$

The first term (the dominant one in the infinite momentum limit) respects scale invariance and we conjecture that the function $\mathcal{M}_0[\lambda,n]$ governing it is the same as in the DNLS equation. In fact we expect that such a model is the ultraviolet limit of our family of theories. An evidence of this is that $\xi_R(\mu)$ flows to 1 for $\xi_R(\mu_0)>1$. Notice that in a Galilean invariant theory correction of order $1/P^2$ are strictly forbidden, while they cannot be excluded in principle here.

We conclude by observing that this brief analysis is consistent with the Galilean invariant model ρ^6 considered at the end of the previous section.

V. CONCLUSION

In conclusion we have extensively studied a family of $1+1$ dimensional theories that describes non-relativistic bosons interacting with a gauge potential: the gauge action has been chosen to be of $B-F$ type plus a "chiral" kinetic term for B. This form was suggested from the dimensional reduction of Chern-Simons theory coupled to non-relativistic matter and it represents a simple way to introduce chiral excitations that can be important in some condensed matter context. The theory was, in fact, proposed as relevant to modelling quantum Hall states: although it was shown in [10] that it fails to achieve one of its goal (the statistical transmutation of bosons on a line), the system presents some very interesting features. First of all, it can be exactly reduced, solving for A_μ and B, to a self-interacting bosonic theory for which a local Lagrangian formulation is possible. Remarkably, the system possesses the non-relativistic scale invariance and for one choice of the parameters we recover an integrable equation (DNLS): for a generic choice we have a scale-invariant perturbation of the integrable model. The one-soliton structure of the theory has been examined, and it exhibits an interesting "chiral" behaviour: one-soliton solutions exist only for a fixed sign of the total momentum and they present a very peculiar particle-like energy-momentum relation inherited from the scale invariance. Solutions with non-trivial boundary conditions at infinity (dark solitons) were also found, existing for the opposite sign of momentum and with finite energy. In this case some care was needed in defining the conserved quantities: in particular we have proposed a definition of the total momentum consistent with the Hamiltonian structure of the theory and leading to a finite result. The scale invariance is broken by our boundary conditions, which entails a complicate dispersion relation. Finally we have studied the quantum dynamics, trying to discuss the relation between the classical one-soliton solutions and the quantum bound-states. The lack of integrability has not allowed a complete solution, but some result were obtained. The two-body problem was solved and the quantum bound states reproduce the chiral behaviour of the classical solutions. Moreover the perturbative computation of the energy for the N-body bound-state was found to be in agreement with the Bohr-Sommerfield quantization of the soliton in the

weak-coupling limit. On the other hands scale invariance was broken at quantum level by a trace anomaly: the fixed point of the renormalization group coincides with the integrable system, while, in the general case, we have deduced the functional form of the bound-states energy by combining the (classical) one-soliton solution with the informations coming from the (quantum) trace anomaly. There remains to prove or, at least, to check our proposal by explicit computations: for example to calculate the functional determinant representing, in the path-integral framework, the non-trivial quantum corrections or to solve explicitly the three-body Schródinger equation. This subject together with the non-abelian extension of the model are currently under investigations.

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